Beating Betz - Energy Extraction Limits in a Uniform Flow Field

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Abstract:

Experiments with diffusers and other flow concentrating devices have shown that the power performance coefficient, C_p, of an energy extraction device defined in relation to the area of flow intercepted at the device may exceed the Betz limit. "Beating Betz" in that sense is long established but no theory has existed to define in a generalised way what ideal limit may apply to C_p in such situations. Recent analysis has resolved this. That analysis is derived in a somewhat different and more general way here. This indicates that, irrespective of the presence of flow concentration systems or other influences that perturb the flow but do not in themselves extract energy, there is a universal ideal limit of energy extraction. This is found to be 8/9 of the upstream kinetic energy in the streamtube associated with the energy extraction. Moreover the familiar Betz equations for power and thrust coefficients can be generalised in a simple way to express this.

Keywords: Betz limit, energy extraction

1 The Lanchester-Betz-Joukowski limit

The title of this section acknowledges van Kuik [1] and an earlier investigation [2] by Bergey. It appears that Lanchester (1915), Betz (1920) and Joukowski (1920) may all have, probably independently and certainly by methods differing in detail, determined the maximum efficiency of an energy extraction device in open flow. Following van Kuik's recommendations, all these sources are acknowledged, whilst for convenience the limit is still referred to as the Betz limit.

In the full generality of the actuator disc concept, the energy extraction device need not be specifically a rotor. However for convenience, the term rotor will often be used in further discussion. Operation of a rotor in the freestream without diffuser or other system influencing the flow will be referred to as "open flow".

As a preliminary to the later discussion, some elementary considerations about the nature of the energy extraction process and also an alternative expression for maximum C_p in open flow are discussed.

The Betz limit is generally derived from the power coefficient, C_p expressed as:

$$C_p = 4a\,(1-a)^2 \qquad (1)$$

where a is the axial induction at the rotor plane.

By differentiation of equation (1) or otherwise, C_p is found to be maximum at the value *16/27* when a = *1/3*.

In the general case in open flow, where a ranges between *0* and *1*, the following relationship may be derived[1] for the

[1] This result is most easily understood looking ahead to equation (9) with C_t =*8/9* for a =*1/3* and $a = a_m$

maximum power performance coefficient, C_{pm} .

$$C_{pm} = 8/9\left(1-a_m\right) \qquad (6)$$

In equation (6) a_m is the value of axial induction that maximises C_p. $\left(1-a_m\right)$ may be regarded as the ratio of the upstream source area to the rotor plane area.

Thus at maximum C_p, the rotor extracts 8/9 of the kinetic energy flowing through the upstream source area.

Equation (6) may seem rather an incidental relationship in the open flow case. However it will be shown that this result is completely general and applies to a rotor in a diffuser or a rotor influenced by other systems such as terrain that perturb the flow field without in themselves extracting energy.

2 Limit relationships of power, thrust and inflow

The open flow actuator disc theory has been established for over 90 years whilst van Bussel [3] in a recent comprehensive review notes that diffuser research has been in progress over 50 years. In the early 1980's, Oman, Gilbert and Foreman [4] conducted experimental work on the DAWT (diffuser augmented wind turbine) concept showing that power coefficients exceeding the Betz limit could be obtained. In 1999, Hansen [5] published CFD results confirming that the Betz limit could be exceeded. Hansen's paper noted that the increase in C_p was in proportion to the augmentation of mass flow achieved by the diffuser but that this did not explicitly define a limit for C_p.

Recent analysis by Jamieson [6] has determined new relationships for limiting values of C_p and a preliminary validation showed close quantitative agreement with Hansen's CFD results [5].

The analysis presented here compliments the work [6] and starts in a similar way but derives the same results in a rather more general way. It also clarifies a critical distinction between the plane at which energy is extracted and a plane where the axial induction is half of that in the far wake. These planes are co-incident in the case of an open flow rotor but not co-incident in general and never in the case of flow concentrator systems.

The conventional definitions of power coefficient, C_p, thrust coefficient, C_t, and axial induction factor, a, are maintained. Thus all three terms are referenced to the undisturbed upstream wind velocity and the applicable area is the area of the energy extraction system that intercepts the flow with a being the axial induction local to the extraction device. Note also that any system that augments local inflow velocity will have negative values of axial induction in some states.

From the basic definitions of the power co-efficient, C_p, and the thrust co-efficient, C_t, the power to thrust ratio can be expressed as in equation (7).

Power: $P = \frac{1}{2}\rho V^3 A_{ext} C_p$

Thrust: $T = \frac{1}{2}\rho V^2 A_{ext} C_t$

$$\frac{P}{T} = V.\frac{C_p}{C_t} \qquad (7)$$

However considering also the basic definition of power as a product of force and velocity as applied at the energy extraction plane;

$$P = T\,V\left(1-a\right)$$

Hence

$$\frac{P}{T} = V\left(1-a\right) \qquad (8)$$

And hence, from equations (7) and (8),

$$\frac{C_p}{C_t} = \left(1-a\right) \qquad (9)$$

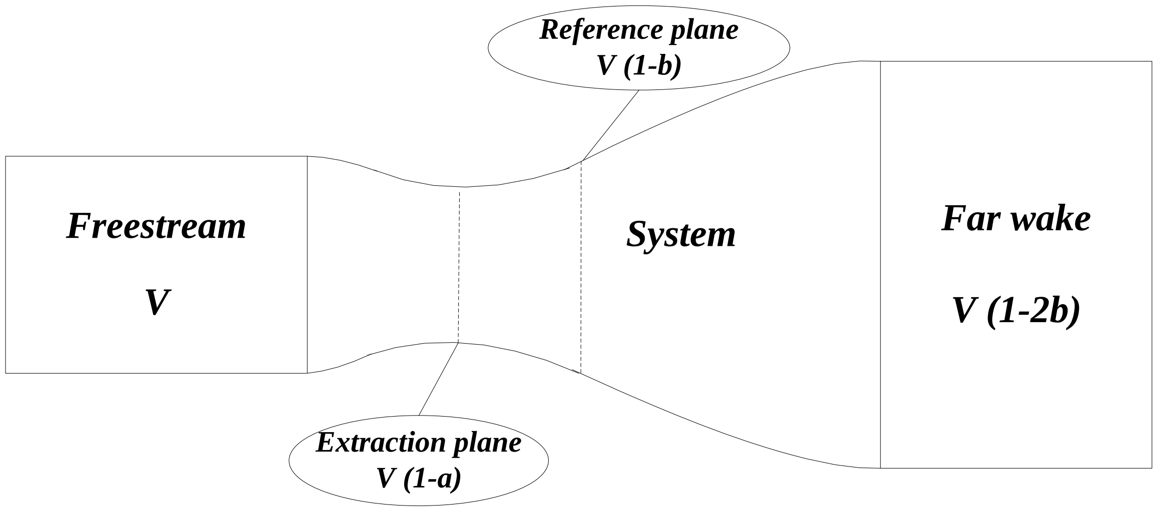

Figure 1 Flow field regions

Equation (9) so derived does not depend on open flow actuator disk theory. It is completely general for any system with a rotor where the local inflow is a fraction $(1-a)$ of the remote undisturbed external wind speed.

A *system* is defined as the region in which axial induction is influenced between the freestream and the far wake. Within the system, two planes are defined (Figure 1):

1. Energy extraction plane of area A_{ext} at which energy extraction takes place, for example by a wind turbine rotor.

2. Reference plane of area A_{ref} where the induction is the same as in open flow and the velocity is V in the absence of purposeful energy extraction or any system energy losses.

The reference plane need not be within the physical body of the system that influences extraction plane but this plane will always exist from continuity of axial flow velocity.

The reference plane may not remain at a fixed location as the level of energy extraction and thrust on the extraction system vary and it need not in principle be downstream of the energy extraction plane.

Denoting axial induction at the reference plane as b, then velocity in the far wake is $V(1-2b)$, a result first determined [1] by R E Froude.

At any plane of area A within the system where there is a pressure difference Δp associated with energy extraction, the thrust, T is given as;

$$T = \Delta p\, A = \frac{1}{2}\rho\, V^2\, AC_t$$

Hence

$$C_t = \frac{2\Delta p}{\rho V^2}$$

Thus the thrust coefficient, C_t, depends only on the pressure difference, Δp , and not on a specific location in the system. (This is not true of C_p).

At the reference plane, conditions are equivalent to open flow and, hence[2];

$$C_t = 4b\,(1-b) \quad (10)$$

[2] It is necessary for this step temporarily to consider energy extraction to be at the reference plane so that the flow process in the diffuser upstream of the reference plane is conservative and immaterial. Thus only the far upstream and far wake conditions need attention allowing the result to be derived as in the standard open flow case.

Denoting axial induction at the energy extraction plane as a and considering conservation of mass in the flow;

$$\rho A_{ext} V(1-a) = \rho A_{ref} V(1-b)$$

In the absence of energy extraction, note that $b = 0$ and let the axial induction at the energy extraction plane be a_0. Then:

$$\rho A_{ext} V(1-a_0) = \rho A_{ref} V$$

Hence

$$1-b = \frac{1-a}{1-a_0}$$

and

$$b = \frac{a-a_0}{1-a_0} \qquad (11)$$

Substituting for b in equation (10) gives;

$$C_t = \frac{4(a-a_0)(1-a)}{(1-a_0)^2} \qquad (12)$$

At this stage it is important to have established that C_t did not depend on a specific location in the system. Only in that case is it justified to interpret equation (12) as in particular defining C_t at the rotor plane enabling the next step.

From equations (9) and (12)

$$C_p = \frac{4(a-a_0)(1-a)^2}{(1-a_0)^2} \qquad (13)$$

Differentiating equation (13) with respect to a determines a maximum at $a = a_m$ of

$$a_m = \frac{1+2a_0}{3} \qquad (14)$$

The associated maximum C_p is then:

$$C_{pm} = \frac{16}{27}(1-a_0) \qquad (15)$$

This can be equivalently expressed as;

$$C_{pm} = \frac{8}{9}(1-a_m) \qquad (16)$$

For the open flow rotor with $a_0 = 0$, equations (12) to (15) correspond, as they must, to the established equations for open flow and the familiar results that $a_m = 1/3$ and $C_{pm} = 16/27$ are evident.

Equation (16) is evidently identical to equation (6) but now established as a general result. By considering, for example, equation (4), $(1-a_m)$ obviously represents the area ratio of upstream source to the rotor plane or equivalently the augmentation of mass flow at the rotor plane compared to open flow through the rotor plane with no energy extraction. This result (equation 16) is highlighted as the most general statement that can be made about maximum energy extraction.

Considering the limit equation (12), on substituting for a_m from equation (8) in equation (6), it is found that:

$$C_t = 8/9 \qquad (17)$$

Whereas a_m and C_{pm} have specific values for each system configuration, this result is now independent of a_0. Equation (17) is therefore a general truth for an optimum energy extraction device in any ideal system configuration. This result was suggested to the author in 1995 by Ken Foreman, as an observed outcome (without theoretical explanation) of his extensive experimental work within Grumman Aerospace in the 1980's with the diffuser augmented wind turbine concept. It was proven in 1991 by van Bussel [7] and now follows directly as a consequence of the generalised limit equations (12) and (14).

Figure 2 illustrates the (C_p, a) plane with the line $C_p = (1-a)$ which is equivalent to $\frac{C_p}{C_t} = (1-a)$ and $C_t = 1$, always bounding the limiting curve. Each C_p curve touches this line at the point where $a = a_c = (\frac{1+a_0}{2})$.

This point is around where, for the open flow rotor, $a = 0.5$, $C_t = 1$, and the rotor enters the turbulent wake state. As may be expected from equation (16) the line $C_p = \frac{8}{9}(1-a)$ intersects each C_p curve at maximum value. Equation (12) will cease to be valid in the turbulent wake state but this does not impede the development of optimal device and system design.

As was observed [6], in the limit state (ideal device in ideal system);

- The design of the device is completely decoupled from design of the system which system is completely characterised by a_0,
- The thrust and thrust coefficient that correspond to optimum rotor loading are independent of the system that includes the rotor and the thrust coefficient is always *8/9*.

Based on experimental results, Phillips [8] observed that a rotor in a real diffuser may be optimally loaded above the ideal value of 8/9 due to the effect of the external flow in assisting wake transport. He considered this finding to conflict with, van Bussel [7] and Hansen [5] Lilley and Rainbird [9], de Vries [10] and by comparison, presumably also the foregoing analysis. However, it is likely that there is no significant conflict.

If the additional mass flow outside diffuser is considered to exert thrust on an associated annular area, the overall system thrust coefficient may still optimally be 8/9.

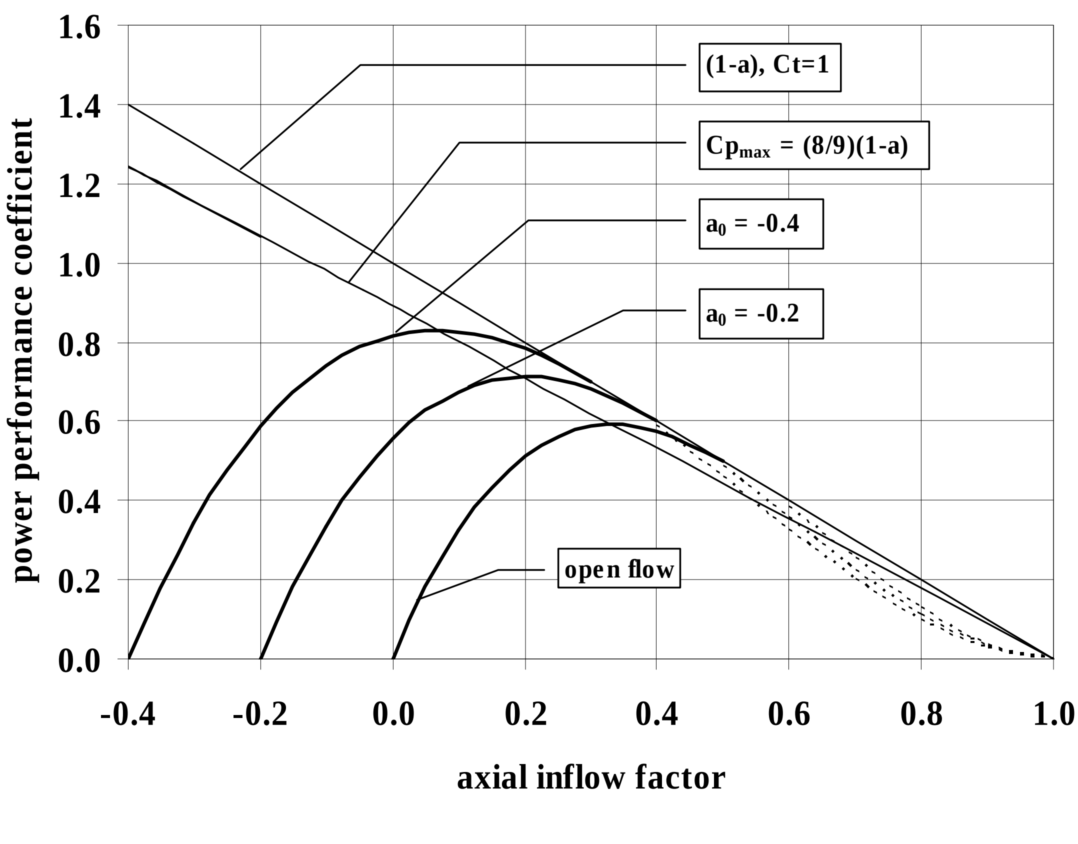

Figure 2 Limiting C_p characteristics

An addition to these conclusions based on equation (16) is that:

- In the ideal limit state, the maximum power performance coefficient is 8/9 of the ratio of flow source area to area at the energy extraction plane or equivalently 8/9 of the mass flow augmentation ratio.

Although this is a speculation and a viscous interaction between the external flow and the wake is beyond the scope of the ideal limit theory, nevertheless, when the system is considered as a whole and all the active mass flow considered, there is not necessarily any violation of equation (17). Foreman [11] obtained experimental results perhaps also with some evidence for an optimal C_t above 8/9 although he had also suggested that the optimal thrust loading was independent of flow augmentation.

It is clearly the case that in any analysis of a real system, the system boundary must encompass all the mass flows playing a role in the energy exchanges.

There is not in general any conflict of the new limit theory with previous work on diffusers although the present work may provide a basis to revisit and re-formulate previous theories and attempts to optimise design of ducts and rotors in ducts. In some previous work "speed up" ratios were introduced. It would now seem better generally to use only one variable, the axial induction defined in the standard way as a velocity reduction fraction.

A generalisation of the blade element momentum equations has been indicated [6] which can directly assist optimisation of ducted rotors. The essence of this generalisation is, following equation (11), to substitute $\frac{a-a_0}{1-a_0}$ for the axial induction in the momentum equations. Although not discussed in [6], there is no requirement in principle to use an average value of a_0 if varying local values of induction (in the rotor absent state) can be specified over the rotor disc area.

The key to the foregoing limit analysis has been the characterisation of the flow in the absence of an energy extraction device by the parameter, a_0. As was discussed further by Jamieson [6], in progressing from ideal limits to real systems a crucial concept is that of an *ideal* system.

3 Generalised energy extraction theory in relation to diffuser design

The new limit theory has significant implications for diffuser design which may be subject of further work. However diffuser design is discussed in this paper primarily because a diffuser CFD experiment was found to be the best immediate source of validation of the limit theory.

A rotor can be optimised and remain in the optimum state by preserving the ratio of rotor speed to wind speed as wind speed changes. Thus the concept of an ideal rotor is straightforward as one which has no drag losses or so-called tip losses. In open flow, the flow field as characterised by the stream tube geometry of the flow through the rotor disc varies with the level of rotor loading.

The same is true in the presence of a diffuser. However the diffuser purposefully constrains the flow field and must therefore be of continuously variable geometry to match the consequent flow field in every state of rotor loading! Only such a diffuser can be considered *ideal* in terms of the limit theory presented.

Corollaries to this are that:

- A fixed geometry (real) diffuser optimised in the absence of rotor loading will not be optimum in the presence of rotor loading.
- Such a diffuser, optimised instead to match the optimum rotor loading state, ($C_t = 8/9$) and thereby maximise output power, will not be optimum in the rotor absent state.

Thus the estimation of a_0 (a characteristic parameter of the *ideal* diffuser) will not be straightforward.

These issues have been addressed [6] and indication provided how to introduce the modelling of rotor and system (diffuser) losses. In summary, in open flow for reasonably well optimised rotors, the system C_t characteristic is independent of rotor efficiency and maximum C_t will correspond to $C_t = 8/9$. For a flow concentration system such as a diffuser, diffuser losses will reduce the optimal level of rotor loading ($C_t < 8/9$) and the maximum attainable performance of the optimally loaded rotor.

In this approach, assuming an intention to design rotor and diffuser reasonably optimally, the analysis of rotor and diffuser efficiency can be usefully de-coupled. In systems, however, with substantial aerodynamic compromises, the coupling will be strong.

Van Bussel [7] predicts the development of a back pressure in relation to the analysis of performance of a real diffuser of fixed geometry. This is a useful concept for diffuser design and reflects the situation where the real diffuser (non-ideal in most states of loading) approaches ideal in a particular loading state. He developed an equation [7] for the rotor C_p of a system with augmented flow as:

$$C_p = 4\,\beta\,\gamma\,\alpha\,(1-b)^2 \qquad (17)$$

The axial induction is denoted as b in equation 17 rather than using a as in the source publication because van Bussel's analysis relates to induction at the reference plane (Figure 1) rather than at the rotor plane. The parameter β is the ratio of diffuser exit area to rotor plane area and γ is an unknown function.

Equation (11) has determined that:

$$b = \frac{a - a_0}{1 - a_0}$$

Substituting in equation (17) for b and comparing with equation (12) gives;

$$(1 - a_0) = \beta\,\gamma$$

Thus although superficially different in form from equation (13), van Bussel's equation (17) is fully consistent with the system of equations (12) to (16). The combination of a real system parameter β with an essentially undetermined system specific parameter γ may understandably have inhibited the realisation that the product has a determinate value for an ideal system.

The exit area ratio, β has a unique value for any given diffuser design and is real in that sense but it is not a characteristic parameter of the "ideal" variable geometry diffuser. Thus γ cannot be determined in general as it is a function of the variable system efficiency of the fixed geometry diffuser that is necessarily specific to each system.

4 Validation of the generalised limit theory

The definition of diffuser efficiency as in Jamieson [6] and as in this paper is new. It is taken to be the ratio of C_t at maximum C_p to the ideal vale of 8/9. It quantifies the extent that a non-optimum diffuser design will reduce the optimal thrust coefficient and hence also the maximum attainable rotor C_p. This definition arose in an attempt to validate the limit theory using CFD results of Hansen [5].

The axial induction factor a, may be eliminated from equations (12) and (13) and C_p expressed in terms of C_t as;

$$C_p = \frac{1}{2} C_t (1 - a_0)\left(1 + \sqrt{1 - C_t}\right) \qquad (18)$$

This form of equation (previously published in [5] and in other sources for the open flow case with $a_0 = 0$) is especially useful for comparison with experimental results as it does not explicitly involve the axial induction factor. The essence of the validation presented [6] was as follows. Hansen [5] quoted an augmentation factor of 1.83 in the absence of energy extraction. The diffuser appeared to optimise in the rotor absent state which gave a direct estimate of a_0 as $a_0 = -0.83$.

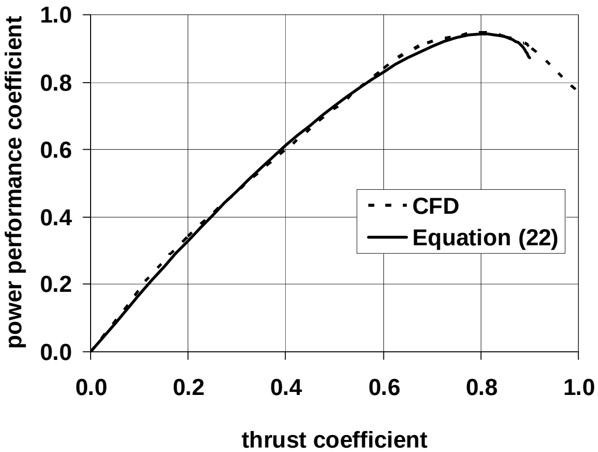

Figure 3 Comparison of theory and CFD

Hansen's simulated rotor is ideal as it achieved the Betz limit in an open flow CFD representation. However the diffuser is not ideal. Maximum C_p occurs (Figure 3) at $C_t = 0.8$ indicating that the diffuser design is not matched to optimal rotor loading for which $C_t = 8/9$.

The effective diffuser efficiency at maximum C_p is:

$$\eta_S = \frac{0.8}{(8/9)} = 0.9 \qquad (19)$$

In a real diffuser, the diffuser efficiency should be a variable function that modifies the apparent axial induction according to how closely the diffuser approaches the optimal performance of an associated ideal diffuser[3]. However previously [6], the diffuser efficiency was treated as constant at 0.9 which is strictly correct only at the critical condition where C_p is maximum. In general if the diffuser is not optimised in the absence of energy extraction the induction in this condition will not be a_0 but instead, $a_0\eta_S$, where η_S is in general a function of a.

Thus C_t is taken as:

$$C_t = \frac{4\left(a - a_0\eta_S\right)\left(1-a\right)}{\left(1-a_0\right)^2} \qquad (20)$$

And from equation (9) for an ideal rotor, C_p is then

$$C_p = \frac{4\left(a - a_0\eta_S\right)\left(1-a\right)^2}{\left(1-a_0\right)^2} \qquad (21)$$

Eliminating a from (19) and (20) :

$$C_p = \frac{1}{2}C_t\left\{\left(1-a_0\eta_S\right) + \left[\left(1-a_0\eta_S\right)^2 - \left(1-a_0\right)^2 C_t\right]^{0.5}\right\} \qquad (22)$$

Equation (22) evidently agrees very well with the CFD data (Figure 3). It is not however correct to treat η_S as a constant and hence equations (21) and (22) will not represent C_p and C_t well although it was argued [6] that this has little effect on the C_p, C_t relationship.

This issue is revisited here in a little more detail. What is required is to model an efficiency term that is unity with the rotor absent when $a = a_0 = -0.83$ but which decreases with increasing a to a value of 0.9 when $C_t = 0.8$ (see equation 19). The simplest representation of this is a linear efficiency variation as:

$$\eta(a) = \frac{a\left(1-\eta_S\right) - a_m + a_0\eta_S}{a_0 - a_m} \qquad (23)$$

Differentiating equation (21) to estimate a_m, leads to equation (24) which is used to evaluate a_m in equation (23).

$$a_m = \frac{1 + 2a_0\eta_S}{3} \qquad (24)$$

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[3] Each point on the real diffuser characteristic may then be considered to represent some ideal diffuser with the particular value of a_0 that corresponds to a characteristic passing through the particular point.

C_t and C_p still retain the same form as in equations (20) and (21) with $\eta(a)$ replacing η_S. The effect of this diffuser efficiency representation on C_p and C_t is now plausible (Figure 4) with a diffuser efficiency of unity at $a = a_0 = -0.83$ and an efficiency of 0.9 when C_p is maximum and $C_t = 0.8$.

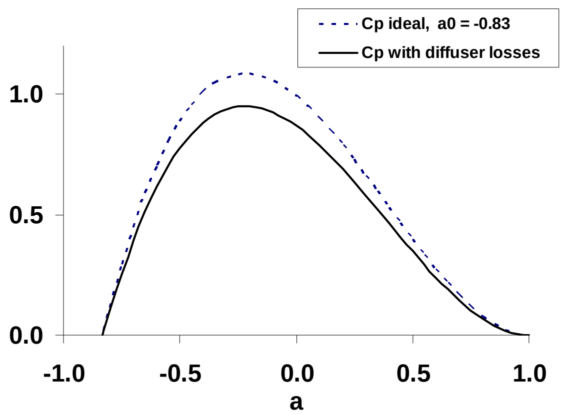

Figure 4 Effect of diffuser inefficiency on power coefficient

Since the diffuser is ideal with rotor absent, then $(a - a_0)$ must be a factor of C_p and C_t and simplifying the expression for $\eta(a)$ in equation (23) after substitution for a_m from equation (24) easily verifies this.

Thus C_t can be represented as:

$$C_t = \frac{4k(a - a_0)(1 - a)}{(1 - a_0)^2} \qquad (25)$$

and also C_p in a corresponding way, where

$$k = \frac{(1 - a_0\eta_S)}{(1 - 3a_0 + 2a_0\eta)} \qquad (26)$$

This leads to a revision of equation (22) as

$$C_p = \frac{1}{2k} C_t (1 - a_0) \left\{ k + \left[k^2 - kC_t \right]^{0.5} \right\} \qquad (27)$$

Comparison of equations (22) and (27) with Hansen's data verify the previous comment [6] that the C_p, C_t relationship is insensitive to the efficiency characteristic if it has an appropriate value at maximum C_p.

Equation (27) is also in good agreement with Hansen's data but not particularly an improvement on equation (22). However, the linear efficiency model is only a more plausibly realistic representation of a diffuser efficiency characteristic than the constant efficiency one and is not physically based beyond that. At present probably only CFD modelling can characterise the efficiency of a particular constant geometry diffuser accurately.

6 Concluding remarks

One dimensional actuator disc theory, long established in the open flow case, has been generalised to provide an ideal limiting theory for rotors operating within any type of system that modifies the flow field but does not in itself extract energy.

This has obvious applicability to wind turbines in diffusers and ducted turbines in all types of energy systems. It may also inform the optimum operation of turbines operating in complex terrain which, like ducts or diffusers, perturb the flow field even in the absence of energy extraction.

The limit theory is derived analytically without any requirement to use empirical information but a very useful validation is obtained comparing with CFD results. Further CFD experiments could certainly provide much more extended and complete validation and address the optimisation of ducts and diffusers. In this context CFD is particularly useful as it can deal with a range of cases from highly idealised (inviscid flow etc.) to thoroughly realistic.

It was previously well established that power coefficients exceeding the Betz limit can be achieved. It appears that whilst 16/27 is not a universal number in actuator disc theory, 8/9 may be. This is both the fraction of the source upstream kinetic energy that can optimally be extracted by a rotor and the thrust coefficient of an optimised rotor regardless of whether it operates in open flow or otherwise.

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